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DYNAMIC INSTABILITY OF TRUNCATED CONICAL SHELLS UNDER PERIODIC AXIAL LOAD

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Abstract—Based on the Dynamic version of Donnell type basic equations neglecting bending deformations before instability, the parametric instability of truncated conical shells subjected to periodic axial load is studied under four different boundary conditions. Applying Galerkin's method, the basic equations are reduced to a system of coupled Mathieu equations, from which the instability regions are determined by using Hsu's results. As a numerical example, instability regions for a completely clamped shell are determined for relatively wide range of frequencies. The effects of static axial load as well as damping force on the instability regions are also examined.

INTRODUCTION

To clarify the parametric instability of conical shells is of great importance for the design of aerospace structures. However, only few researches have been made on this subject under pulsating external pressure. That is, Aulfutov and Razumeev[1] obtained solutions of approximate nature for shells with small cone angles under the assumption of inextensional vibrations. Later, Kornecki[2] analyzed the problem for simply supported conical shells by applying Galerkin's method to the Donnell type basic equations and obtained one-term approximate solutions for only the principal instability regions.

In this paper, the parametric instability of truncated conical shells subjected to periodic axial load is studied under four sets of boundary conditions. The analysis is based on the dynamic version of Donnell type basic equations neglecting bending deformations prior to instability. Applying Galerkin's method with the solution expressed by the product of coordinate functions and unknown time-dependent ones, the basic equations are reduced to a system of coupled Mathieu equations for the time-dependent functions. Normalizing this system, the instability regions are determined by using Hsu's results [3].

As a numerical example, the regions of combination resonance and principal instability for a completely clamped shell under periodic axial load are determined for relatively wide range of frequencies. The effects of static axial load as well as viscous damping force on the instability regions are also investigated.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

We will consider a truncated conical shell with thickness h, slant length l, base circle radius r and semi-vertex angle α, subjected to the uniformly distributed edge forces as shown in Fig. 1, with the resultant axial load

$$P(t) = P_0 + P_t \cos \omega t \tag{1}$$

where P_0 and $P_t \cos \omega t$ are static and pulsating axial loads, respectively.

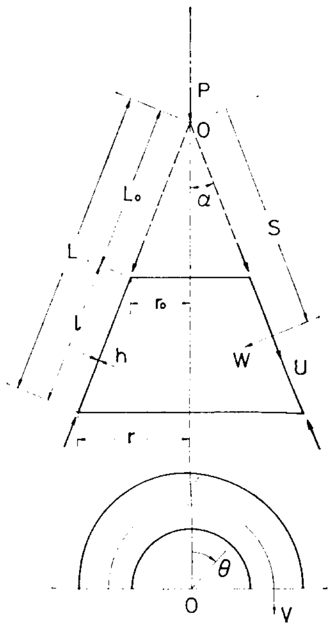

Fig. 1. Coordinates and nomenclatures of the truncated conical shell.

Assuming that the deformation before instability can be represented by a membrane state of stress and that the in-plane inertia forces can be neglected for low frequencies with predominent flexural type, the Donnell type basic equation for dynamic instability may be given by following equations in nondimensional form [2]:

$$\nabla^4 f + \frac{1}{x} w_{,xx} = 0 \tag{2}$$

$$L(w) \equiv \nabla^4 w - 12Z^2 \frac{1}{x} f_{,xx} + (k_{co} + k_{ct} \cos \Omega\tau) \frac{1}{x} w_{,xx} + \kappa w_{,\tau} + w_{,\tau\tau} = 0 \tag{3}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial \phi^2}, \qquad \phi = \theta \sin\alpha \tag{4}$$

and subscripts following a comma stand for partial differentiation.

The relations between the incremental membrane forces and the stress function are given by

$$n_s = \frac{1}{x} f_{,x} + \frac{1}{x^2} f_{,\phi\phi}, \qquad n_\theta = f_{,xx}, \qquad n_{s\theta} = -\left(\frac{1}{x} f_{,\phi}\right)_{,x} \tag{5}$$

while the relations between the incremental displacements and membrance forces are given by

$$\left.\begin{array}{l} u_{,x} = n_s - \nu n_\theta, \qquad \frac{1}{x}(u + v_{,\phi} - w) = n_\theta - \nu n_s \\ \frac{1}{x}(u_{,\phi} - v + x v_{,x}) = 2(1+\nu) n_{s\theta} \end{array}\right\}. \tag{6}$$

In the foregoing equations, the nondimensional notations are related to the physical variables as

$$\left.\begin{aligned} &x = \frac{s}{L}, \quad \tau = \frac{t}{L^2}\sqrt{\left(\frac{D}{\mu h}\right)}, \quad (u, v) = \frac{\tan\alpha}{h}(U, V), \quad w = \frac{W}{h} \\ &(n_s, n_\theta, n_{s\theta}) = \frac{L\tan\alpha}{Eh^2}(N_s, N_\theta, N_{s\theta}), \quad f = \frac{F\tan\alpha}{Eh^2 L} \\ &\gamma = \frac{L_0}{L}, \quad Z = \frac{L\sqrt{1-\nu^2}}{h\tan\alpha}, \quad (k_{ct}, k_{c0}) = \frac{L}{\pi D\sin 2\alpha}(P_t, P_0) \\ &\kappa = \frac{cL^2}{\sqrt{\mu h D}}, \quad \Omega = \omega L^2\sqrt{\frac{\mu h}{D}} \end{aligned}\right\}. \tag{7}$$

In these expressions, $D = Eh^3/12(1-\nu^2)$ is the flexural rigidity of the shell, E, ν and μ are Young's modulus, Poisson's ratio and mass density, respectively, while c is the viscous damping coefficient. Z and γ are a shape factor and a truncation ratio, respectively.

Concerning the boundary conditions for the instability problem, the following four cases will be considered at $x = \gamma$ and 1.

$$\left.\begin{aligned} &C1: w = w_{,x} = u = v = 0 \\ &C2: w = w_{,x} = n_s = n_{s\theta} = 0 \\ &S1: w = w_{,xx} + \frac{\nu}{x}w_{,x} = u = v = 0 \\ &S2: w = w_{,xx} + \frac{\nu}{x}w_{,x} = n_s = n_{s\theta} = 0 \end{aligned}\right\}. \tag{8}$$

In these expressions, C and S correspond to the clamped and simply supported cases, respectively, while numerals 1 and 2 specify the in-plane boundary conditions.

The problem consists in finding the limiting values of Ω for which the basic equations have unbounded solutions under the given loading and boundary conditions.

METHOD OF SOLUTION

The solution w of equations (2) and (3) will be assumed as

$$\left.\begin{aligned} &w = \sum_m g_m(\tau)w_m(x)\cos N\theta \qquad (m = 1, 2, \ldots) \\ &w_m(x) = \sum_{j=0}^{4} c_{mj}x^{m+j-1} \end{aligned}\right\} \tag{9}$$

where N (=integer) is the number of circumferential waves, $g_m(\tau)$ is the unknown time-dependent function and $c_{mj}(j = 0 \sim 4)$ are constants chosen so that w_m satisfies the boundary conditions for w, i.e., S or C, by taking c_{m0} as unity.†

Substituting equation (9) into equation (2) and considering equations (5) and (6), the stress function f can be determined so as to satisfy the remaining in-plane boundary conditions in equations (8). With these expressions for w and f, we apply Galerkin's method to

† The same coordinate function $w_m(x)$ has been used by the author for the analyses of static buckling [4] and free transverse vibration[5] of conical shells.

equation (3), that is

$$\int_0^{\pi}\int_{\gamma}^{1} L(w)\sum_{k=0}^{4} c_{nk}\, x^{n+k}\cos N\theta\,\mathrm{d}x\,\mathrm{d}\theta = 0 \qquad (n = 1, 2, \ldots) \tag{10}$$

which finally leads to the following system of coupled Matheiu equations in $g_m(\tau)$.

$$\sum_m \{R_{mn} g_{m,\tau\tau} + \kappa R_{mn} g_{m,\tau} + [S_{mn} + (k_{co} + k_{ct}\cos\Omega\tau)T_{mn}]g_m\} = 0. \tag{11}$$

$$(m, n = 1, 2, \ldots)$$

In these equations, R_{mn}, S_{mn} and T_{mn} are coefficients dependent on ν, Z, γ, N and the boundary conditions, the actual expressions of which are not given here to save the space. It is to be noted that these coefficients are all symmetric with respect to their subscripts m and n, that is, $R_{mn} = R_{nm}$ etc.

Equations (11) represent those for static buckling[4] and free transverse vibration[5] of conical shells as special cases.

In order to apply Hsu's results, the natural frequencies $\Omega_p (p = 1, 2, \ldots)$ and the corresponding modes G_p of free transverse vibration of the conical shell under static axial load are determined by putting $\kappa = k_{ct} = 0$ in equation (11). Normalizing equation (11) with these results, we have

$$G_{p,\tau\tau} + \Omega_p{}^2 G_p = k_{ct}\cos\Omega\tau\sum_q \lambda_{pq} G_q - \kappa\sum_q \xi_{pq} G_{q,\tau} \tag{12}$$

$$(p, q = 1, 2, \ldots)$$

where λ_{pq} and $\xi_{pq}(>0)$ are coefficients symmetrical in p and q, respectively.

Approximate solutions for the parametric instability of these equations have been obtained by Hsu[3]. According to his results, we have the stability boundaries for the combination resonance associated with ith and jth modes of vibration as .

$$\Omega_{ij} = \Omega_i + \Omega_j \pm \frac{\xi_{ii} + \xi_{jj}}{2}\left(\frac{E_{ij}{}^2 - \kappa^2\xi_{ii}\xi_{jj}}{\xi_{ii}\xi_{jj}}\right)^{1.2}‡ \tag{13}$$

where

$$E_{ij} = k_{ct}\lambda_{ij}/2(\Omega_i\Omega_j)^{1/2} \tag{14}$$

No combination resonance will occur in the vicinity of $|\Omega_i - \Omega_j|$, as $\lambda_{pq} - \lambda_{qp}$. The so-called principal instability regions are obtained from equation (13) by putting $i = j$. Equation (13) is applicable to the case of small damping with $E_{ij}{}^2 \geqq \kappa^2\xi_{ii}\xi_{jj}$. For the case when damping is absent, equation (13) is reduced to

$$\Omega_{ij} = \Omega_i + \Omega_j \pm E_{ij} \tag{15}$$

When the values for ν, Z, γ, κ, k_{co}, k_{ct} and the boundary conditions are given, all the instability regions (first approximation) can be determined for prescribed range of frequency ω by repeating the foregoing procedure, varying the wave number N successively.

‡ According to Bolotin's first approximation[6], the principal instability regions for equations (12) are given by

$$\Omega_{ij} = 2\Omega_i\left[1 \pm \frac{1}{\Omega_i}(E_{ii}{}^2 - \kappa^2\xi_{ii}{}^2)^{1/2}\right]^{1/2}$$

which will be seen to be practically the same as those given in the foregoing, for small amplitudes of exciting forces.

NUMERICAL EXAMPLE

As a numerical example, the instability regions are determined for a completely clamped ($C1$) conical shell with the following data:

$$\alpha = 20°, \quad \gamma_0 = 2{\cdot}13 \text{ in}, \quad \gamma = 4{\cdot}87 \text{ in}, \quad h = 0{\cdot}02 \text{ in}$$

$$E = 3 \times 10^7 \text{ psi}, \quad \mu = 0{\cdot}776 \times 10^{-3} \text{ lb sec}^2/\text{in}^4 \quad Z = 1860, \quad \gamma = 0{\cdot}438.$$

The lowest three natural frequencies for various values of N are calculated, taking eight terms of unknown time-dependent function g_m into consideration. The results are given in Fig. 2. The same results have been obtained in[5] and ascertained to be in good agreement with the experimental results obtained by Seide[7]. The lowest natural frequency occurs at $N = 6$ with $\Omega = 1087$, i.e. $\omega/2\pi = 1{\cdot}017$ (KH_z). The static critical axial load is given by $k_c = 1291$, i.e. $P = 4020$ (lb) with $N = 9$ [4].

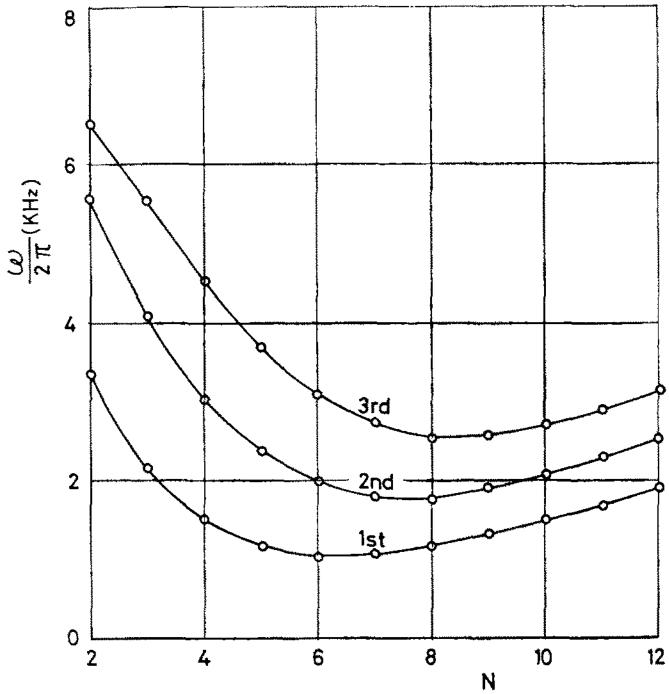

Fig. 2. Natural frequencies for perfectly clamped case.

Instabilibity regions

Varying the amplitude of periodic axial load up to 10 per cent of the aforementioned buckling load, the boundaries of the instability regions with no damping and no static axial load are determined for the frequency range covering four times the lowest natural frequency. The results are illustrated in Fig. 3. In this figure, the instability regions are shown by shaded areas. Upper numerals in parentheses indicate wave number N and lower ones two modes of free vibration excited in the combination resonance. The cases with the same lower numerals correspond to the regions of principal instability. It will be seen that the regions of principal instability are much wider than those of combination resonance.

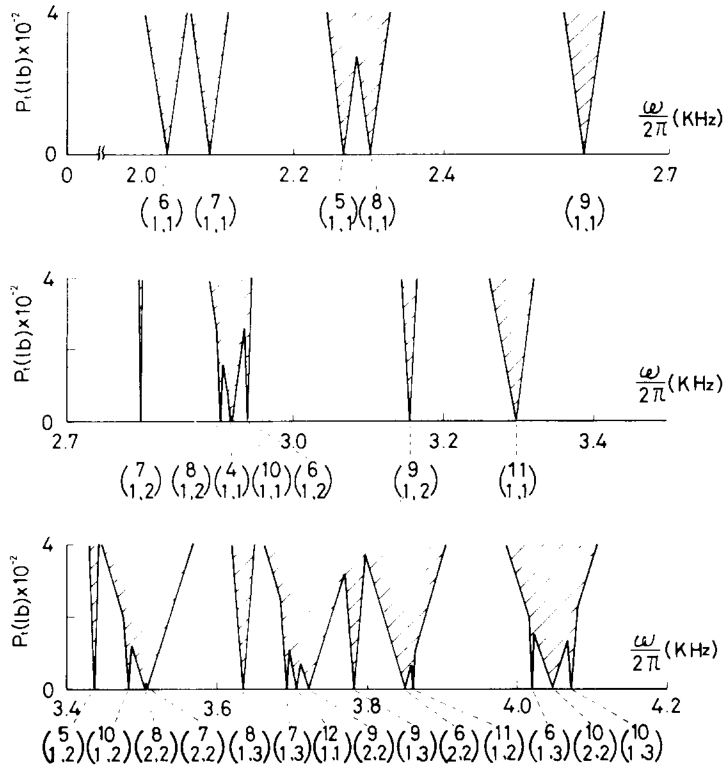

Fig. 3. Instability regions of only periodical axial load. ($P_0 = 0$, $c = 0$).

Effects of static axial load

Choosing the static axial load as 5 and 10 per cent of the static buckling load and setting damping equal to zero, the instability regions in the neighborhood of $\Omega = 2\Omega_1$, and $\Omega = \Omega_1 + \Omega_2$ with $N = 9$, are determined, with the results as given in Fig. 4. It will be observed

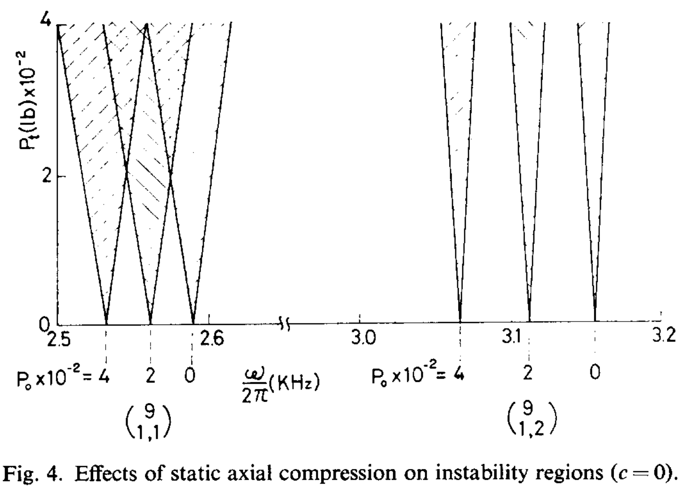

Fig. 4. Effects of static axial compression on instability regions ($c = 0$).

that as the static compressive force is increased, the instability regions shift almost parallel towards the lower frequency regions.

Effects of damping

Setting static axial load equal to zero, the instability regions with viscous damping are determined in the neighborhood of $\Omega = 2\Omega_1$, and $\Omega = \Omega_1 + \Omega_2$ with $N = 9$. The results are shown in Fig. 5. It will be seen that the presence of damping reduces the instability regions such that for small periodic axial load, the parametric instability is not possible. It is to be noted that the effect of damping is more pronounced for the cases of combination resonance.

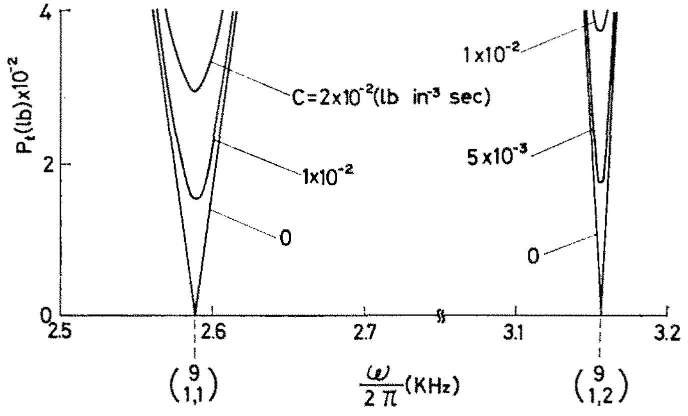

Fig. 5. Effects of viscous damping on instability regions ($P_0 = 0$).

CONCLUSION

Approximate solutions are obtained for the parametric instability of truncated conical shells subjected to periodic axial load, under four different boundary conditions. Numerical results are given for a completely clamped shell. Main results obtained are summarized as follows:

(1) The relative openness of the regions of combination resonance is much narrower than that of the regions of principal instability.

(2) The instability regions are moved toward the lower frequencies as a whole owing to the static axial compressive force.

(3) The presence of damping reduces the instability regions.

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Абстракт — На основе динамического варианта основных уравнений типа Доннелла, пренебрегающих деформации изгиба до момента потери устойчивости, исследуется параметрическая неустойчивость усеченных конических оболочек, нагруженных периодической осебой нагрузкой, для четырех разных граничных условий. Применяя метод Галеркина, сводятся основные уравнения к системе сопряженных уравнений Матье, из которых определяются районы неустойчивости, используя результаты Су. В качестве примера определяются районы неустойчивости полно защемленной оболочки, для широкого круга частот. Обсуждаются, также, эффекты как статической осебой нагрузки так и демифирующей силы, на районы неустойчивости.